Schröder–Bernstein & Comments thereon

Based on notes for MT232P

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1 Schröder–Bernstein: Statement and Proof

Theorem 1.1. If $f : X \to Y$ and $g : Y \to X$ are both injective for sets X and Y, then X and Y are numerically equivalent.

The idea is to partition X into $X_\infty \cap X_x \cap X_y$, and Y into $Y_\infty \cap Y_x \cap Y_y$.

Maps $X_\infty \to Y_\infty$, $X_x \to Y_y$ and $X_y \to Y_x$ may then be established as bijections.

Definition 1.1. For $x \in X$, the *x-ancestry* of x is the sequence, so long as it is defined, given by $\{x, g^{-1}(x), f^{-1}(g^{-1}(x)), \dots\}$.
Similarly the *y-ancestry* of y for $y \in Y$ is given by $\{y, f^{-1}(y), g^{-1}(f^{-1}(y)), \dots\}$

Note that the ancestry of a number may have only one element.
If for example, $x \in X \setminus g(Y)$, the *x-ancestry* of x is just the finite sequence $\{x\}$. Ancestries may be infinitely long, in which case they may or may not repeat after some time. Also note that a number $a \in X \cap Y$ has both an *x-ancestry* and a *y-ancestry*, although this is not relevant for the proof.

Now the partitioning sets may be defined.

- Let X_∞ be the set of all $x \in X$ with infinite *x-ancestry*.
- Let X_x be the set of all $x \in X$ with finite *x-ancestry* with odd length. That is to say, the final element of the sequence is in the set $X \setminus g(Y)$.
- Let X_y be the set of all $x \in X$ with finite *x-ancestry* with even length. That is to say, the final element of the sequence is in the set $Y \setminus f(X)$.
- Let Y_∞ be the set of all $y \in Y$ with infinite *y-ancestry*.
- Let Y_x be the set of all $y \in Y$ with finite *y-ancestry* with even length. That is to say, the final element of the sequence is in the set $X \setminus g(Y)$.
- Let Y_y be the set of all $y \in Y$ with finite *y-ancestry* with even length. That is to say, the final element of the sequence is in the set $Y \setminus f(X)$.

Now we may define bijections.

- $f : X_\infty \to Y_\infty$ is a surjection, because $\forall y \in Y_\infty$, y has an immediate ancestor $f^{-1}(y) = x$, so $\exists f(x) = y$. It's also trivially injective because f is injective.
- $f : X_x \to Y_x$ is a surjection, because $\forall y \in Y_x$, y has an immediate ancestor $f^{-1}(y) = x$, so $\exists f(x) = y$. It's also trivially injective because f is injective.
- $g^{-1} : X_y \to Y_y$ is a surjection, because $\forall y \in Y_y$, $g(y)$ has *x-ancestry* $\{g(y), y, f^{-1}(y), \dots\}$, which is just $g(y)$ appended to the *y-ancestry* of y, and so $g(y) \in X_y$. It follows that $\forall y \in Y_y$, $\exists x \in X_y$ s.t. $g^{-1}(x) = y$. g^{-1} is also an injection because $g^{-1}(x_1) = g^{-1}(x_2) \implies g(g^{-1}(x_1)) = g(g^{-1}(x_2)) \implies x_1 = x_2$

Finally , let $h(x) : X \mapsto Y$ be defined like so:

$$h(x) = \begin{cases} f(x) & x \in X_\infty \cup X_x \\ g^{-1}(x) & x \in X_y \end{cases}$$

Because X_∞, X_x and X_y form a partition, and each forms a bijection, the function as a whole forms a bijection.

2 Comments

The Schröder–Bernstein theorem stands out as one of the harder proofs from the MT232P module (Intro to analysis), likely due to the uniqueness of the proof structure as compared to a standard contradiction/induction proof that would otherwise be employed when beginning pure maths. The theorem is also interesting in that, depending on the context, it can seem obviously true or very confusing. An alternate equivalent statement of the theorem is the following:

Theorem 2.1. If a set X is numerically equivalent to a subset of the set Y, and the set Y is numerically equivalent to a subset of the set X, then X is numerically equivalent to Y.

After being introduced to uncountability of the reals, power set hierarchies, and sets containing themselves, it did not seem obvious at all that this statement held. It seemed likely, even, that there were two obtuse sets which could each contain and be in bijection with parts of the other. Once you've been introduced to cardinalities and the orderings thereon, the statement reads like so:

Theorem 2.2. For two sets X and Y, $(|X| \leq |Y|) \wedge (|Y| \leq |X|) \implies (|X| = |Y|)$

This statement draws a direct analogy to the reals and, when read in this form, seems absolutely obvious. This does not help, however, because the proof of this statement is not so obvious at all.

While I was taking the course, I found it quite difficult to write the proof in any satisfactory way which felt both rigorous and concise. This is due, in large part, to an implicit yet unreasonable assumption made in many of the proof statements: $X \cap Y = \{\}$

This assumption does not need to be made for the theorem to hold, but it's nonetheless the intuitive picture upon which the proof is based. The real idea behind X_x and Y_x is that the ancestries halt in the set X. But of course, if may halt in an element which features in both sets. What to do then? In my proof above I have separated the idea of ancestry into *x-ancestry* and *y-ancestry* to explicitly make apparent the idea that a number does not necessarily have a unique ancestry. This is also why the X_x is defined based on odd-length ancestry, rather than on ancestry which halts in X.