The uncountability of the reals, Schröder-Bernstein proof

Proof using the mapping between the natural power set and the unit interval

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1 The uncountability of the reals

Theorem 1.1. There is no bijection between $\mathbb{R}$ and $\mathbb{N}$

An injection will be established from the $\mathcal{P}(\mathbb{N})$ to $[0,1]$. An injection will be assumed from $[0,1]$ to $\mathbb{N}$. This will imply the existence of an injection from $\mathcal{P}(\mathbb{N})$ to $\mathbb{N}$, which will be shown to contradict Cantor's theorem. This contradiction will imply the theorem.

Define $g : \mathcal{P}(\mathbb{N}) \to [0,1]$, where $g(p)$ is the number in $[0,1]$ whose n-th digit after the decimal point is 7 if $n \in p$ and is 3 otherwise. Note that all $x \in g(\mathcal{P}(\mathbb{N}))$ have each only 1 decimal expansion.

For some $p \in \mathcal{P}(\mathbb{N})$, for every $n \in \mathbb{N}$, $n \in p \iff$ the n-th digit of $g(p)$ is 7. If $g(p) = g(q)$ for some q, then they have the same expansion, and so have 7s in the same places, and so $p = q$. g is therfore an injection.

Assume, for the sake of contradiction, that there is an injection $f : [0,1] \to \mathbb{N}$. It follows that $(f \circ g) : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$ is an injection.

Define $h : \mathbb{N} \to \mathcal{P}(\mathbb{N})$, where $h(n) = \{n\}$. If $h(n) = h(m)$, then $\{n\} = \{m\}$ and so $n = m$. h is therefore an injection.

The existence of the injections $h : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ and $(f \circ g) : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$ implies, by Schröder-Bernstein, that there is a bijection between $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$. This contradicts Cantor's theorem. By *reductio ad absurdum*, there is no injection $f : [0,1] \to \mathbb{N}$. Any bijection between $\mathbb{R}$ and $\mathbb{N}$ would immediately permit an injection from $[0,1]$ to $\mathbb{N}$, and so no such bijection exists. QED.

2 Comments

I found a version of this proof on stack exchange in November, and I immediately took to it. I abhor the numerical diagonal argument as a proof due to how fiddly it is. You have to care so much about the repeating $0.\ldots 9999\ldots$ case. You also have to manually construct the number which is not present in your counting of the reals, a cumbersome task which you already would have done when proving cantor's theorem. (Which is much easier). In general I don't like arguments involving decimal or other expansions, because we never showed that a number may be written in such a way. However, given the fact that we were clearly allowed to use decimal expansions in the exam, I opted to use this method, because it does a lot of heavy lifting and only takes up 6 lines. It's also pleasing to use the other results we'd proved to do something.